10/20/21

Calculus III

Last Time: Fubini's Thm

- If f is cts on $R = [a,b] \times [c,d]$,

then $\int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy = \iint_R f(x,y)\,dA$

$$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$$

Ex: Compute $\iint_R \frac{x}{1+xy}\,dA$ on $R = [0,1] \times [0,1]$.

Sol 1: $\iint_R \frac{x}{1+xy}\,dA = \int_{x=0}^{1} \int_{y=0}^{1} \frac{x}{1+xy}\,dy\,dx$

Inner Integral: $\int_{y=0}^{1} \frac{x}{1+xy}\,dy$ $\qquad u(y) = 1+xy$, $du = x\,dy$

$$= \int_{y=0}^{1} \frac{1}{1+xy} \cdot x\,dy$$

$$= \int_{y=0}^{1} \frac{1}{u}\,du$$

$$= \ln|u| \Big|_{y=0}^{1}$$

$$= \ln|1+xy| \Big|_{y=0}^{1}$$

$$= \ln(1+x) - \ln(1+0)$$

$$= \ln(1+x)$$

Outer Integral: $\int_{x=0}^{1} \ln(1+x)\,dx = \left[x\ln(1+x) - \int \frac{x}{1+x}\,dx\right]_{x=0}^{1}$ $\qquad u = \ln(1+x)$, $du = \frac{1}{1+x}\,dx$, $dv = dx$, $v = x$

$$= \left[x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx \right]_{x=0}^{1}$$

$$= \left[x\ln(1+x) - \left(x - \ln(1+x)\right) \right]_{x=0}^{1}$$

$$= \left(\ln(2) - (1 - \ln(2))\right) - \left(0 - (0 - \ln(1))\right)$$

$$= 2\ln(2) - 1$$

Inner Integral: $\int_{x=0}^{1} \frac{x}{\underbrace{1+xy}_{\text{hard}}} \, \cancel{dx}$

In principle: could try $\left(\begin{aligned} u(x) &= 1+xy \\ \cancel{du} &\cancel{= y\,dx} \end{aligned} \right)$ $\quad x = \frac{u-1}{y}$

$$= \int_{x=0}^{1} \frac{\left(\frac{u-1}{y}\right)}{u} \cdot \frac{1}{y} \, du$$

$$= \underbrace{\frac{1}{y^2}}_{\substack{\text{contributes} \\ \text{to discontinuity} \\ \text{if } y=0}} \int_{x=0}^{1} \frac{u-1}{u} \, du$$

ABANDON

Exercise: Compare $\iint_R ye^{-xy}\,dA$ on $R = [0,2]\times[0,3]$

Write out both possible orders of integration ...

Point: Sometimes one order is more computable than another order.

Def^n: The average value of function $f(x,y)$ on region R is $\frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

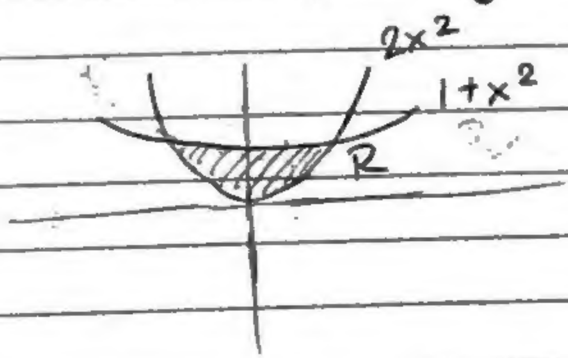
Goal: integrate over more than just rectangles...

Ex: Compute the (net) volume of the solid bounded by $y=2x^2$, $y=1+x^2$, and $z=x+2y$ and $z=0$.

Net volume is $\iint_R ((x+2y)-0)\,dA$

over $R = \left\{ (x,y) : (x,y) \text{ between } y=2x^2 \text{ and } y=1+x^2 \right\}$

Now a picture of region in xy-plane:



For fixed x, we know $2x^2 \le y \le 1+x^2$.

To find x-bounds solve $\left(\begin{array}{c} 2x^2 = 1+x^2 \\ \text{iff } x = \pm 1 \end{array} \right)$

$\therefore R = \{(x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$

Thus, because our parameterization of R is ___ ,
we can write our double integral as an iterated integral!

Sol: We just saw $R = \{(x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$

$$\therefore \iint_R (x+2y)\,dA = \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{x=-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{1+x^2} dx$$

$$= \int_{x=-1}^{1} \Big[\left(x(1+x^2) + (1+x^2)^2\right) + \left(x(2x^2) + (2x^2)^2\right)\Big] dx$$

$$= \int_{x=-1}^{1} \left(x(1+x^2-2x^2) + \left((1+x^2)^2 - (2x^2)^2\right)\right) dx$$

$$= \int_{x=-1}^{1} \left(x(1-x^2) + (1+x^2+2x^2)(1+x^2-2x^2)\right) dx$$

$$= \int_{x=-1}^{1} (1+x+3x^2)(1-x^2)\,dx$$

$$= \int_{x=-1}^{1} (1+x+2x^2-x^3-3x^4)\,dx$$

$$= \left[x + \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{3}{5}x^5\right]_{x=-1}^{1}$$

$$= \left(1+\frac{1}{2}+\frac{2}{3}-\frac{1}{4}-\frac{3}{5}\right)-\left(-1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}-\frac{3}{5}\right)$$

$$= \frac{32}{15}$$

TAKEAWAY: If R is parameterized by something like:

$$R = \{(x,y) : c_1 \le x \le c_2 ,\ g_1(x) \le y \le g_2(x)\},$$

then $\iint_R f(x,y)\,dx = \int_{x=c_1}^{c_2}\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Similarly, $R = \{(x,y) : c_1 \le y \le c_2 ,\ g_1(y) \le x \le g_2(y)\}$

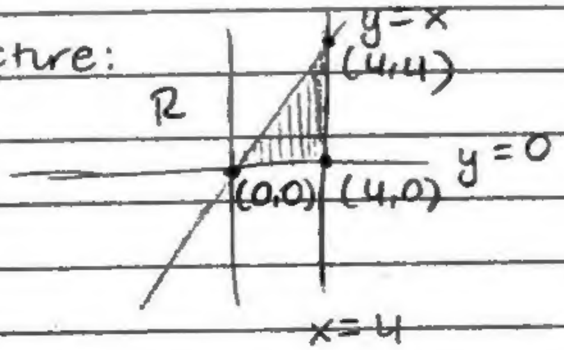
yields $\iint_R f(x,y)\,dA = \int_{y=c_1}^{c_2}\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$

Ex: Compute $\iint_R y^2 e^{xy}\,dA$ for R bounded by

$y=x,\ y=0,\ x=4.$

$R = \{(x,y) : 0 \le y \le 4,\ y \le x \le 4\}$

Picture:



$R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le x\}$

Sol: $\iint_R y^2 e^{xy}\, dA = \int_{y=0}^{4}\int_{x=y}^{4} y^2 e^{xy}\, dx\, dy$

Easier

$$= \int_{x=0}^{4}\int_{y=0}^{x} y^2 e^{xy}\, dy\, dx$$

$$\iint_R y^2 e^{xy}\, dA = \int_{y=0}^{4}\int_{x=y}^{4} y^2 e^{xy}\, dx\, dy$$

Inner: $\int_{x=y}^{4} y^2 e^{xy}\, dx = \int_{x=y}^{4} y e^{xy}\, y\, dx$

$$= \int_{x=y}^{4} y e^{u}\, du \qquad \begin{aligned} u &= xy \\ du &= y\,dx \end{aligned}$$

$$= y\left[e^{u}\right]_{x=y}^{u}$$

$$= y\left[e^{xy}\right]_{x=y}^{4}$$

$$= y\left(e^{4y} - e^{y^2}\right)$$

$$= ye^{4y} - ye^{y^2}$$

$$\therefore \iint_R y^2 e^{xy} = \int_{y=0}^{4} \left(ye^{4y} - ye^{y^2}\right) dy$$

$$= \int_{y=0}^{4} ye^{4y}\, dy - \int_{y=0}^{4} ye^{y^2}\, dy$$

Integration by parts

$u = y$
$du = dy$
$dv = e^{4y}\, du$
$v = \frac{1}{4} e^{4y}$

$$= \left[\frac{1}{4} ye^{4y} - \int \frac{1}{4} e^{4y}\, dy - \frac{1}{2}\int e^{u}\, du\right]_{y=0}^{4}$$

$$= \left[\frac{1}{4} y e^{4y} - \frac{1}{16} e^{4y} - \frac{1}{2} e^{y^2} \right]_{y=0}^{4}$$

$$= \left(e^{16} - \frac{1}{6} e^{16} - \frac{1}{2} e^{16} \right)$$

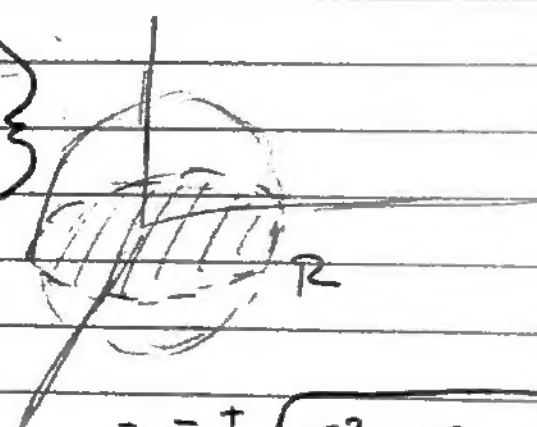
$$= \frac{1}{16}(1 - e^{16}) + \frac{1}{2} - \frac{1}{2} e^{16}$$

Motivating Question: What is the volume of the sphere?

Set up:

$S = \{(x,y,z) : x^2+y^2+z^2=r^2\}$

$R = \{(x,y) : \cancel{-r \leq x \leq r},\ -\sqrt{r^2-x^2} \leq y \leq \sqrt{r^2-x^2}\}$



We need to parameterize this region

$z = \pm\sqrt{r^2-x^2-y^2}$